

The propagation of weak concentration discontinuities in the isothermal flow of a multicomponent mixture in a porous medium with phase transitions[☆]

A.V. Koldoba, Ye.V. Koldoba

Moscow, Russia

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Abstract

It is shown that for the general case of a system of non-linear equations, describing multicomponent isothermal flow in a porous medium with phase transitions, as in hyperbolic systems, weak concentration discontinuities propagate with finite velocities, which are determined by solving an eigenvalue problem. If the seeping phases are incompressible and there are no phase transitions, the results obtained for weak discontinuities transfer into the well-known formulae for the Buckley – Leverett model. The results are demonstrated for the case of two-component seepage with phase transitions.

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Multicomponent and multiphase isothermal flow in a porous medium with intensive interphase mass exchange is described by a system of non-linear partial differential equations, to solve which the presence of concentration distributions is characteristic.^{1–6} This system of equations can be simplified in the following two cases: 1) if the phases are incompressible, there are no phase transitions and the capillary pressure jump can be neglected (the flow is described by the Buckley – Leverett model, the properties of which are well-known), and 2) if the phases are incompressible, a phase transition occurs and Amagat's law is satisfied (the volume of the mixture does not change when a phase transition occurs).

However, Amagat's law and the Buckley – Leverett model are far from always applicable. For example, when modelling gassed oil the evolved gas cannot be assumed to be incompressible. Also, if no special relation between the density of the mixture and the composition and pressure is assumed, the system of equations cannot be reduced to classical equations (hyperbolic, parabolic, etc.).

1. Multicomponent flow

We will consider a model of the seepage of an M -phase N -component mixture, which is widely used for solving problems of predicting the development of oil-gas-containing strata.^{1–6} For low seepage rates, local thermodynamic equilibrium can be established by fairly intensive mass transfer in each elementary volume. Of course, by fixing the temperature T , and the total concentration c_i of each i -th component of the mixture one can establish a number of

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E-mail address: koldoba@keldysh.ru (A.V. Koldoba).

phases in which the mixture becomes stratified, their molar concentrations in each phase $c_{i\alpha}$, the molar densities n_α and the volume saturation s_α ; here and henceforth the subscript i denotes the number of the component, while the subscript α is the number of the phase. The seepage rate of each phase is given by Darcy’s law

$$W_\alpha = -K \frac{k_\alpha}{\eta_\alpha} \frac{\partial p}{\partial x} \tag{1.1}$$

where K is the absolute permeability, η_α is the viscosity and $k_\alpha(s_\alpha)$ is the relative permeability. Capillary effects are only taken into account by introducing relative permeabilities, ignoring effects related to the presence of the interphase boundary (adsorption) and its curvature (the shift in pressure and phase equilibrium due to capillarity).

The total seepage rate is defined as

$$W = \sum_\alpha W_\alpha = -KB \frac{\partial p}{\partial x}, \quad B = \sum_\alpha \frac{k_\alpha}{\eta_\alpha}$$

where B is the total mobility; the quantity $\varphi_\alpha = k_\alpha / (B\eta_\alpha)$ is the fraction of the α -th phase in the flow: $W_\alpha = \varphi_\alpha W$.

Since the number of moles of each component is conserved, for each of them we can write the transport equation

$$m \partial n_i / \partial t + \partial Q_i / \partial x = 0, \quad i = 1, \dots, N \tag{1.2}$$

where m is the porosity of the layer, n_i is the molar density of the i -th component (the number of moles of the i -th component per unit volume of the porous space) and Q_i is the density of the molar flow of the i -th component. The quantities n_i and Q_i characterize the mixture as a whole and are expressed in terms of the phase characteristics as follows:

$$n_i = \sum_\alpha c_{i\alpha} n_\alpha s_\alpha, \quad Q_i = \sum_\alpha c_{i\alpha} n_\alpha W_\alpha \tag{1.3}$$

where n_α is the molar density of the α -th phase and $c_{i\alpha}$ is the molar concentration of the i -th component in the α -th phase, which is defined in terms of $N_{i\alpha}$ – the number of moles of the i -th component in the α -th phase:

$$c_{i\alpha} = N_{i\alpha} \left(\sum_k N_{k\alpha} \right)^{-1}$$

Substituting expressions (1.1) and (1.3) into Eq. (1.2) we obtain non-linear equations for the flow of a multicomponent mixture

$$m \frac{\partial}{\partial t} \left(\sum_\alpha c_{i\alpha} n_\alpha s_\alpha \right) - \frac{\partial}{\partial x} \left(\sum_\alpha c_{i\alpha} n_\alpha K \frac{k_\alpha}{\eta_\alpha} \frac{\partial p}{\partial x} \right) = 0, \quad i = 1, \dots, N \tag{1.4}$$

These equations must be supplemented by the phase-equilibrium relations (the equality of the chemical potentials and pressures in the phases)

$$\mu_{i,1} = \mu_{i,2} = \dots, \quad p_1 = p_2 = \dots = p$$

Moreover, it follows from the definitions of the phase saturation and molar concentration of the i -th component in the α -th phase that

$$\sum_\alpha s_\alpha = 1, \quad \sum_i c_{i\alpha} = 1$$

Note that in system (1.4), which describes N -component flow, there are N equations and N independent variables (the pressure p and $N - 1$ concentrations c_1, c_2, \dots, c_{N-1}).

2. The propagation of weak discontinuities

The system of Eq. (1.2) allows of discontinuous solutions. The Hugoniot relations at a discontinuity are written in the standard form

$$-mU[n_i] + [Q_i] = 0 \quad (2.1)$$

where U is the velocity of the discontinuity.

We will investigate the behaviour of weak concentration discontinuities in the general case (without assuming any dependence of the mixture density on the composition and pressure). Suppose on a certain line $x = x(t)$ the concentrations undergo a weak discontinuity, i.e. along this line the quantities c_i and, respectively, Q_i and n_i are continuous, and their derivatives undergo a discontinuity. Differentiating the continuous functions $c_k(t, x)$ along the lines $x = x(t)$, we obtain

$$0 = [dc_k/dt] = [\partial c_k/\partial t] + \dot{x}[\partial c_k/\partial x] \quad (2.2)$$

Note that, since the quantities n_i , Q_i , k_α and η_α are continuous, we obtain from relations (2.1) and (1.1) that the pressure has no weak discontinuity:

$$[\partial p/\partial x] = 0$$

Then, by relation (1.1), the phase flow rates are also continuous. Summing Eq. (1.2) over i , we obtain

$$m\partial n/\partial t + \partial Q/\partial x = 0 \quad (2.3)$$

where $n = \sum_i n_i$ is the total density of the mixture and $Q = \sum_i Q_i = \sum_\alpha n_\alpha W_\alpha$ is the total flow of the mixture.

We will use the relations

$$n_j = c_j n, \quad Q_j = \chi_j Q, \quad W_* = Q/(mn)$$

$$\sum_i \chi_i = 1, \quad \chi_i = \sum_\alpha c_{i\alpha} n_\alpha k_\alpha \eta_\alpha \left(\sum_\alpha n_\alpha k_\alpha \eta_\alpha \right)^{-1} \quad (2.4)$$

Here χ_i is the fraction of the i -th component in the flow, where W_* is the effective flow rate of the mixture. We express $\partial Q/\partial x$ from Eq. (2.3) and substitute it into Eq. (1.2). We then take into account the fact that $n = n(p, c_1, \dots, c_{N-1})$. We obtain

$$\frac{\partial c_i}{\partial t} + (c_i - \chi_i) \sum_k \left(\frac{\partial \ln n}{\partial c_k} \right)_p \frac{\partial c_k}{\partial t} + (c_i - \chi_i) \left(\frac{\partial \ln n}{\partial p} \right)_c \frac{\partial p}{\partial t} + W_* \frac{\partial \chi_i}{\partial x} = 0 \quad (2.5)$$

Subtracting one of the expressions of (2.5), written on different sides of the line $x = x(t)$, from the other and taking the continuity of the function W_* into account, we have

$$\sum_k B_{ik} [\partial c_k/\partial t] + W_* \sum_k A_{ik} [\partial c_k/\partial x] = 0 \quad (2.6)$$

where

$$A_{ik} = (\partial \chi_i / \partial c_k)_p, \quad B_{ik} = \delta_{ik} + (c_i - \chi_i) (\partial \ln n / \partial c_k)_p \quad (2.7)$$

Combining relations (2.6) and (2.2), we conclude that the quantities $[\partial c_k/\partial x]$ and the velocities of weak discontinuities \dot{x} are connected by the relations

$$W_* \sum_k A_{ik} [\partial c_k/\partial x] = -\dot{x} \sum_k B_{ik} [\partial c_k/\partial x] \quad (2.8)$$

Hence, the velocities of weak discontinuities are determined in the form $\dot{x} = W_* \lambda$, where λ are the eigenvalues of the problem

$$\mathbf{A}\mathbf{r} = \lambda \mathbf{B}\mathbf{r}$$

where **A** and **B** are matrices with elements (2.7). Here the quantities $[\partial c_k / \partial x]$ are proportional to the components of the corresponding right eigenvectors.

It follows from the normalization condition

$$\sum_i c_i = 1$$

that the functions $\chi_i = \chi_i(c_1, c_2, \dots)$, defined by formula (2.4), can be converted to homogeneous functions of zero degree in the variable c_i , by making the replacement $c_i = c_i \left(\sum_k c_k \right)^{-1}$. It then follows from Euler's theorem on homogeneous functions that

$$\sum_k (\partial \chi_i / \partial c_k)_p c_k = 0 \tag{2.9}$$

One of the solutions of the eigenvalue problem (2.8) has the form

$$\dot{x} = 0, \quad [\partial c_k / \partial x] = c_k \tag{2.10}$$

It is obvious that this solution has no physical meaning since $c_k \geq 0$. On the other hand, when $\dot{x} \neq 0$, summing the quantities (2.7) over i taking into account (2.10) and the first equation of (2.4), we obtain

$$\sum_k [\partial c_k / \partial x] = 0$$

The additional relation obtained denotes that, for the equations of the flow of an N -component mixture, there are no more than $N - 1$ non-zero propagation velocities of weak concentration discontinuities. Hence, it follows from the non-linear equations of multicomponent flow that weak concentration discontinuities propagate with finite velocities, which are given by formula (2.7), and there are no more than $N - 1$. It remains to emphasize that weak discontinuities propagate along certain curves which, however, cannot be called characteristics. In classical hyperbolic systems, not only weak discontinuities propagate along characteristics, but certain characteristic relations are also satisfied.

3. Two-component flow

We will illustrate the results obtained above using the example of the simple but important practical case of a two-component mixture.

We will denote the concentration of the more volatile (second) component by $c_2 = c$, in which case the concentration of the first component will be $c_1 = 1 - c$. The two-component mixture can exist in a single-phase or two-phase state. The more dense phase will be assumed to be a liquid and we will denote the quantities corresponding to it by the subscript l , and the components of the less-dense phase (gaseous) will be denoted by the subscript g . We will denote the volume saturation of the gas phase by $s_g = s$, and that of the liquid phase by $s_l = 1 - s$. The fraction of the gas phase (the Buckley – Leverett function) and of the liquid phase in the flow

$$\varphi_g = \varphi, \quad \varphi_l = 1 - \varphi; \quad \varphi = \frac{k_g / \eta_g}{k_g / \eta_g + k_l / \eta_l}$$

The fraction of the first and second components in the flow

$$\chi_2 = \chi, \quad \chi_1 = 1 - \chi, \quad \chi = \frac{c_g n_g k_g / \eta_g + c_l n_l k_l / \eta_l}{n_g k_g / \eta_g + n_l k_l / \eta_l} = \frac{c_g n_g \varphi + c_l n_l (1 - \varphi)}{n_g \varphi + n_l (1 - \varphi)}$$

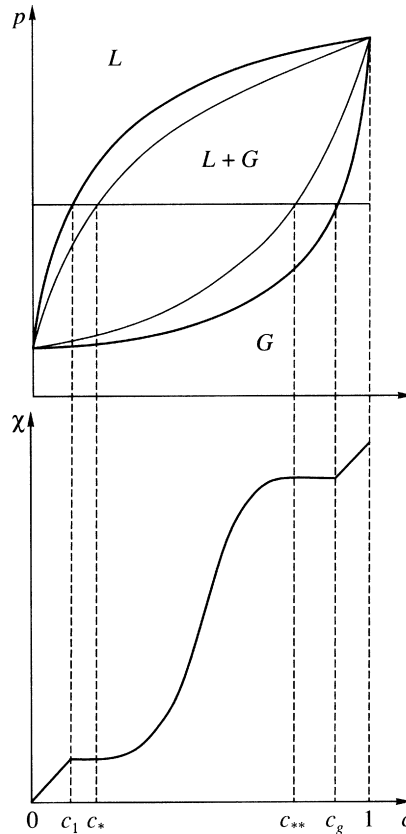


Fig. 1.

In this notation, the equations of two-component flow take the form

$$m \frac{\partial}{\partial t} (c_g n_g s + c_l n_l (1 - s)) + \frac{\partial}{\partial x} (n_g c_g \varphi + n_l c_l (1 - \varphi)) W = 0$$

$$m \frac{\partial}{\partial t} (n_g s + n_l (1 - s)) + \frac{\partial}{\partial x} (n_g \varphi + n_l (1 - \varphi)) W = 0$$
(3.1)

Here $n_g = n_g(p, c)$, $n_l = n_l(p, c)$ are the molar densities of the gas and liquid phases, and $c_g = c_g(p, c)$, $c_l = c_l(p, c)$ are the molar concentrations of the gas and liquid phases.

Since, there is no more than one propagation velocity of weak discontinuities for a two-component mixture (as was shown above), to calculate the physically admissible velocity it is sufficient to consider one of the equations of system (2.8) (for example, the second):

$$W_* \left(\frac{\partial \chi}{\partial c} \right)_p \left[\frac{\partial c}{\partial x} \right] = \dot{x} \left(1 + (c - \chi) \left(\frac{\partial \ln n}{\partial c} \right)_p \right) \left[\frac{\partial c}{\partial x} \right]; \quad \frac{\partial}{\partial c} = \frac{\partial}{\partial c_2} - \frac{\partial}{\partial c_1}$$

Hence, we obtain for the velocity of the weak discontinuity

$$\dot{x} = W_* \left(\frac{\partial \chi}{\partial c} \right)_p \left[1 + (c - \chi) \left(\frac{\partial \ln n}{\partial c} \right)_p \right]^{-1}$$
(3.2)

A graph of the function $\chi(c)$ for a fixed pressure is shown in the lower part of Fig. 1. In the upper part we show a typical phase diagram of the binary mixture. The heavy curves represent the limits of the two-phase region (the boiling and condensation curves – the phase-equilibrium curves), which specify the equilibrium concentrations in the liquid and gaseous phases as a function of the pressure; the thin curves represent the boundaries of connectedness (mobility) of

the phases, due to the residual saturation. In the region of single-phase flow (when $0 \leq c \leq c_l$ or $c_g \leq c \leq 1$) we have $\chi(c) = c$. In the two-phase region, when one of the phases is not connected, i.e. when $c_l \leq c \leq c^*$ or $c^{**} \leq c \leq c_g$, we have $\chi = \text{const}$. In the two-phase region, where both phases are mobile (i.e. when $c^* \leq c \leq c^{**}$) the function $\chi(c)$ has a characteristic S-shaped form.

For the case of incompressible phases, formulae (3.2) are simplified. In the region of single-phase flow we have

$$\chi = c, \quad W_* = W/m, \quad \dot{x} = W/m$$

In the region of two-phase flow ($L + G$), if the phases are incompressible and there are no phase transitions, we obtain the well known relation defining the propagation velocity of weak discontinuities in the Buckley – Leverett model.

$$\dot{x} = (W/m) \partial \varphi / \partial s$$

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